

# Field equations and vector order parameter in braneworld applications.\*

Boris E. Meierovich

*P.L.Kapitza Institute for Physical Problems.*

*2 Kosygina street, Moscow 119334, Russia<sup>†</sup>*

(Dated: September 27, 2010)

## Abstract

Phase transitions with spontaneous symmetry breaking and vector order parameter are considered in multidimensional theory of general relativity. Covariant equations, describing the gravitational properties of topological defects, are derived. The topological defects are classified in accordance with the symmetry of the covariant derivative of the vector order parameter. The abilities of the derived equations are demonstrated in application to the brane world concept. New solutions of the Einstein equations with a transverse vector order parameter are presented. In the vicinity of phase transition the solutions are found analytically. Comparison with the commonly used scalar multiplet approach demonstrates the advantages of the vector order parameter.

PACS numbers: 04.50.-h, 98.80.Cq

---

\* Text of the plenary report by B.E.Meierovich at the International Conference "Modern problems of gravitation, cosmology, and relativistic astrophysics", Moscow, June 27 – July 3, 2010 (RUDN-10).

<sup>†</sup>URL: <http://www.kapitza.ras.ru/people/meierovich/Welcome.html>

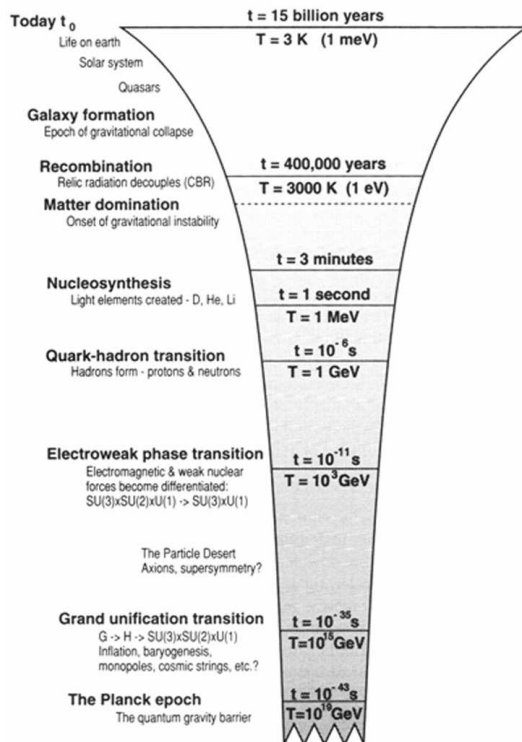


FIG. 1: Universe evolution as a sequence of phase transitions [3].

## I. INTRODUCTION

Macroscopic Landau theory of phase transitions with changes of symmetry was initially developed for crystals [1]. Cosmological implication of phase transitions was initiated by Kirzhnits [2], and modern standard cosmology is actually a sequence of phase transitions with spontaneous symmetry breaking (grand unification, electro-weak, quark-hadron, ...), see Figure 1, taken from the book [3] by Vilenkin and Shellard. The macroscopic theory of phase transitions allows to consider gravitational properties of topological defects (appearing in the state of lower symmetry) self-consistently, even without the knowledge of the nature of physical vacuum.

The theories of braneworld and multidimensional gravity continue numerous attempts to apply the spontaneous symmetry breaking for clarifying the origin of long-standing problems in physics, such as enormous hierarchy of energy and mass scales, dark matter and dark energy effects, and others. In particular, the braneworld is considered as a topological defect, inevitably accompanying the phase transitions with spontaneous symmetry breaking. Topological defects (domain walls, strings, monopoles, textures,...) are described mathematically

by the order parameter, introduced by Landau [1].

Following the pioneer works of Polyakov [4] and 't Hooft [5], the specific topological defects - monopoles - were generally described by the multiple sets of scalar fields. In braneworld theories the hedgehog-type configurations of scalar multiplets in extra dimensions played the role of the order parameters (see [6] and references there in).

Though the scalar multiplet model is self-consistent, it is not the direct way for generalization of a plane monopole to the curved space-time. Contrary to the flat geometry, in a curved space-time scalar multiplets and real vectors are transformed differently. In application to the braneworld with two extra dimensions the spontaneous symmetry breaking with a vector order parameter was considered in [7]. The comparison of the two approaches showed great advantage of the vector order parameter over the scalar multiplets. The equations were of less order and more simple. Their solutions had additional parametric freedom, that allowed the existence of the braneworld without any restrictions of fine-tuning type. Particular derivations in [7] were performed for the special case of two extra dimensions, diagonal metric tensor, and dependence on the only one coordinate - the distance from the brane.

In view of the evident advantages of the vector order parameter, and having in mind its numerous possible applications, it is worth deriving general covariant equations describing gravitational properties of topological defects. It is the main goal of this presentation. The result is summarized in section VI The abilities of the derived equations are demonstrated in application to the braneworld concept. In particular, new solutions of the Einstein equations with a transverse vector order parameter are presented in section VII B In the vicinity of phase transition the solutions are found analytically. Properties of a neutral quantum particle in the braneworld metric are considered in section VIII Section IX contains concluding remarks.

## II. SYMMETRIC AND ANTISYMMETRIC DERIVATIVE OF THE VECTOR ORDER PARAMETER

Let  $\phi_I$  be a vector order parameter. Its covariant derivative  $\phi_{I;K}$  can be presented as a sum of a symmetric  $G_{IK}$  and an antisymmetric  $F_{IK}$  terms:

$$\phi_{I;K} = G_{IK} + F_{IK}, \quad G_{IK} = \frac{1}{2}(\phi_{I;K} + \phi_{K;I}), \quad F_{IK} = \frac{1}{2}(\phi_{I;K} - \phi_{K;I}). \quad (1)$$

In general relativity the order parameter enters the Lagrangian via scalar bilinear combinations of its covariant derivatives and via a scalar potential  $V$  allowing the spontaneous symmetry breaking. A bilinear combination of the covariant derivatives is a 4-index tensor:  $S_{IKLM} = \phi_{I;K}\phi_{L;M}$ . The most general form of the scalar  $S$ , formed via contractions of  $S_{IKLM}$ , is

$$S = (ag^{IK}g^{LM} + bg^{IL}g^{KM} + cg^{IM}g^{KL}) S_{IKLM}, \quad (2)$$

where  $a, b$ , and  $c$  are arbitrary constants. The classification of topological defects with vector order parameters is most convenient in terms of the symmetric and antisymmetric parts of  $\phi_{I;K}$ . In view of  $G_K^L F_L^K = 0$ , and  $F_{M;K} F^{K;M} = -F_{M;K} F^{M;K}$ , the scalar (2) can be presented in the form

$$S = a (G_K^K)^2 + (b + c) G_K^L G_L^K + (b - c) F_K^L F_L^K. \quad (3)$$

The last term with antisymmetric derivatives is identical to electromagnetism. It becomes clear in the common notations  $A_I = \phi_I/2$ ,  $F_{IK} = A_{I;K} - A_{K;I}$ . The bilinear combination of the derivatives  $F_{IK} F^{IK}$  is the same as in electrodynamics. In view of the symmetry of Christoffel symbols  $\Gamma_{IK}^L = \Gamma_{KI}^L$ ,

$$A_{I;K} - A_{K;I} = \frac{\partial A_I}{\partial x^K} - \frac{\partial A_K}{\partial x^I},$$

and the combination  $F_{IK} F^{IK}$  is free of the derivatives of the metric tensor. On the contrary, the two first terms in (3) with symmetric covariant derivatives contain not only the components of the metric tensor  $g^{IK}$ , but also the derivatives  $\frac{\partial g_{IK}}{\partial x^L}$ . The difference between the two terms with symmetric tensors is caused by the curvature of space-time, see (12) below.

## III. LAGRANGIAN

In the notations

$$a = A, \quad b + c = B, \quad b - c = C \quad (4)$$

the Lagrangian of a topological defect is

$$L\left(\phi_I, \frac{\partial\phi_I}{\partial x^K}, g^{IK}, \frac{\partial g_{IK}}{\partial x^L}\right) = A(G_M^M)^2 + BG_{MN}G^{MN} + CF_{MN}F^{MN} - V(\phi_M\phi^M). \quad (5)$$

Dealing with variational derivatives, it is convenient to express all terms in (5) as functions of  $\phi_I, \frac{\partial\phi_I}{\partial x^K}, g^{IK}, \frac{\partial g_{IK}}{\partial x^L}$ :

$$\phi^K = g^{IK}\phi_I, \quad \phi_{I;K} = \frac{\partial\phi_I}{\partial x^K} - \Gamma_{IK}^L\phi_L, \quad \Gamma_{IK}^L = \frac{1}{2}g^{LM}\left(\frac{\partial g_{MI}}{\partial x^K} + \frac{\partial g_{MK}}{\partial x^I} - \frac{\partial g_{IK}}{\partial x^M}\right), \dots$$

#### IV. VECTOR FIELD EQUATIONS

The vector field  $\phi_I$  obeys the Euler-Lagrange equations

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^L}\left(\sqrt{-g}\frac{\partial L}{\partial\frac{\partial\phi_I}{\partial x^L}}\right) = \frac{\partial L}{\partial\phi_I}. \quad (6)$$

Here and below  $\sqrt{-g}$  stands for  $\sqrt{(-1)^{D-1}g}$ , where  $D$  is the dimension of space-time. In terms of  $a, b$ , and  $c$  the variational derivatives  $\frac{\partial L}{\partial\frac{\partial\phi_I}{\partial x^L}}$  and  $\frac{\partial L}{\partial\phi_I}$  are

$$\frac{\partial L}{\partial\frac{\partial\phi_I}{\partial x^L}} = 2\left(ag^{IL}\phi_{;K}^K + b\phi^{I;L} + c\phi^{L;I}\right), \quad \frac{\partial L}{\partial\phi_I} = \left(ag^{NK}g^{LM} + bg^{NL}g^{KM} + cg^{NM}g^{KL}\right)\frac{\partial}{\partial\phi_I}\phi_{N;K}\phi_{L;M} - \frac{\partial}{\partial\phi_I}V(g^{NK}\phi_N\phi_K).$$

In a locally geodesic reference system (where the Christoffel's symbols together with the derivatives  $\frac{\partial g_{IK}}{\partial x^L}$  are zeros)  $\frac{\partial}{\partial\phi_I}\phi_{N;K}\phi_{L;M} = 0$ , and  $\frac{\partial L}{\partial\phi_I} = -2V'\phi^I$ . Here and below

$$V' = \frac{dV}{d(\phi_L\phi^L)}. \quad (7)$$

The vector field equations (6), having a covariant form

$$a\phi_{;L;I}^L + b\phi_{I;L}^L + c\phi_{;I;L}^L = -V'\phi_I \quad (8)$$

in a locally geodesic system, remain the same in all other reference frames. In terms of  $A$ ,  $B$ , and  $C$

$$AG_{L;I}^L + BG_{I;L}^L - CF_{I;L}^L = -V'\phi_I. \quad (9)$$

There are two independent terms with the symmetric tensor  $G$  in (9) and one with the antisymmetric tensor  $F$ . The physical difference between the two symmetric terms becomes clear if we set  $F_{IK} = 0$ . Then (9) reduces to

$$A\phi_{;L;I}^L + B\phi_{I;L}^L = -V'\phi_I \quad (10)$$

and the two left terms differ by the order of differentiation. In general relativity the second covariant derivatives are not invariant against the replacement of the order of differentiation:

$$A^I_{;K;L} - A^I_{;L;K} = -R^I_{MKL}A^M. \quad (11)$$

The difference between the two terms in (10) is caused by the curvature of space-time:

$$\phi^L_{;I;L} - \phi^L_{;L;I} = -R^L_{KIL}\phi^K = R_{IK}\phi^K. \quad (12)$$

In the flat space-time the Ricci tensor  $R_{IK} = R^L_{KLI} = -R^L_{KIL} = 0$ , and in case  $F_{IK} = 0$  there is no physical difference between the two first terms in (8):

$$a\phi^L_{;L;I} + b\phi^L_{;I;L} = (a+b)\phi^L_{;I;L}, \quad R_{IK} = 0.$$

## V. ENERGY-MOMENTUM TENSOR

In a locally geodesic coordinate system the energy-momentum tensor can be written in a covariant form as follows:

$$T_{IK} = -g_{IK}L + 2\frac{\partial L}{\partial g^{IK}} + 2g_{MI}g_{NK}\left(\frac{\partial L}{\partial \frac{\partial g_{MN}}{\partial x^L}}\right)_{;L}. \quad (13)$$

In terms of symmetric and antisymmetric tensors (1) we find:

$$\frac{\partial L}{\partial g^{IK}} = 2(AG^L_L G_{IK} + BG^L_K G_{IL} + CF^L_I F_{KL}) - V'\phi_I\phi_K, \quad (14)$$

$$\frac{\partial L}{\partial \frac{\partial g_{MN}}{\partial x^L}} = -A\phi^P_{;P}(g^{LN}\phi^M + g^{LM}\phi^N - g^{NM}\phi^L) - B(G^{LN}\phi^M + G^{LM}\phi^N - G^{MN}\phi^L). \quad (15)$$

The tensor (15) is presented in a symmetric form against the indexes  $M, N$ . Substituting (14) and (15) into (13), we find the following covariant expression for the energy-momentum tensor:

$$\begin{aligned} T_{IK} = & -g_{IK}L + 2V'\phi_K\phi_I + 2Ag_{IK}(G^M_M\phi^L)_{;L} + 2B\left[(G_{IK}\phi^L)_{;L} - G^L_K F_{IL} - G^L_I F_{KL}\right] \\ & + 2C(2F^L_I F_{LK} - F^L_{K;L}\phi_I - F^L_{I;L}\phi_K). \end{aligned} \quad (16)$$

The vector field equations (9) were used to reduce  $T_{IK}$  to a rather simple form (16).

To confirm the correctness of the energy-momentum tensor (16) it is necessary to demonstrate that the covariant divergence  $T^K_{I;K}$  is zero. Using the vector field equations (9), I exclude  $V_{;I} = 2V'\phi^L\phi_{L;I}$  and  $(V'\phi^K\phi_I)_{;K}$ , and present  $T^K_{I;K}$  as

$$T^K_{I;K} = Aa_I + Bb_I + Cc_I. \quad (17)$$

The coefficients  $A, B$ , and  $C$  are arbitrary constants. However, it doesn't mean that all three vectors in (17) are zeros separately. The vectors  $a_I, b_I$ , and  $c_I$  can be reduced to simple forms:

$$a_I = 2 (\phi_{;K;I}^K \phi_L - \phi_{;K;L}^K \phi_I)^{:L}, \quad b_I = 2 (G_{I;K}^K \phi_L - G_{L;K}^K \phi_I)^{:L}, \quad c_I = 2 (F_{L;K}^K \phi_I - F_{I;K}^K \phi_L)^{:L}. \quad (18)$$

The covariant divergence of the energy-momentum tensor (17) with  $a_I, b_I$ , and  $c_I$ , given by (18), is evidently zero due to the vector field equations (9). For details of derivations see [8].

## VI. COVARIANT EQUATIONS

If a topological defect, associated with a vector order parameter, plays the dominant role in the formation of the space-time structure, then the gravitational properties of the system are determined by the joint set of vector field equations (9) and Einstein equations

$$R_{IK} - \frac{1}{2} g_{IK} R = \kappa^2 T_{IK} \quad (19)$$

with  $T_{IK}$  (16).  $\kappa^2$  is the (multidimensional) gravitational constant. Vector field equations are not independent. They follow from Einstein equations due to Bianchi identities. In practice it is convenient to use a combination of vector field and Einstein equations. In general there are three arbitrary constants  $A, B$ , and  $C$ . Particular physical cases can be classified by their relations.

## VII. APPLICATIONS TO BRANEWORLD CONCEPT

From the point of view of macroscopic theory of phase transitions the brane is a topological defect, inevitably accompanying the spontaneous symmetry breaking. The order parameter was traditionally considered as a hedgehog-type multiplet of scalar fields (see [6] and references there in) and as a longitudinal spacelike vector ( $\phi_K \phi^K < 0$ ) [7] directed along and depending on the same specified coordinate – direction from the brane hypersurface. In the case of two extra dimensions the whole ( $D = d_0 + 2$ )-dimensional space-time has the structure  $M^{d_0} \times R^1 \times \Phi^1$  and the metric

$$ds^2 = e^{2\gamma(l)} \eta_{\mu\nu} dx^\mu dx^\nu - (dl^2 + e^{2\beta(l)} d\varphi^2). \quad (20)$$

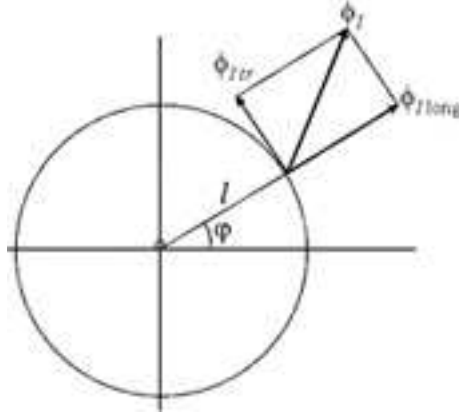


FIG. 2: Plane of extra dimensions  $l$  and  $\varphi$ .

Here  $\eta_{\mu\nu}$  is the flat  $d_0$ -dimensional Minkovsky brane metric ( $\mu, \nu = 0, 1, \dots, d_0 - 1$ ,  $d_0 > 1$ ), and  $\varphi$  is the angular cylindrical extradimensional coordinate.  $\gamma$  and  $\beta$  are functions of the distinguished extradimensional coordinate  $x^{d_0} = l$  – the distance from the center, i.e. from the brane.  $e^{\beta(l)} = r(l)$  is the circular radius. Greek indices  $\mu, \nu, \dots$  correspond to  $d_0$ -dimensional space-time on the brane, and  $I, K, \dots$  – to all  $D = d_0 + 2$  coordinates. The metric tensor  $g_{IK}$  is diagonal. The curvature of the metric on brane due to the matter is supposed to be much smaller than the curvature of the bulk caused by the brane formation.

Vector order parameter, located in two extra dimensions, can be directed arbitrary, and not only along the distance from the brane, see Figure 2. The two cases, when the vector is directed along and across the distance from the brane, are considered separately below.

### A. Symmetric covariant derivative. Longitudinal vector field

In the case of longitudinal vector field (when  $\phi_I = \phi(l) \delta_{I d_0}$  is directed along and depends upon the same coordinate  $l$ ), the covariant derivative  $\phi_{I;K}$  is a symmetric tensor:  $F_{IK} = 0$ ,  $G_{IK} = \phi_{I;K} = \phi_{K;I}$ . The set of covariant equations

$$\begin{aligned} R_{IK} - \frac{1}{2} g_{IK} R &= \kappa^2 \left[ 2V' \phi_K \phi_I + 2A g_{IK} (\phi_{;M}^M \phi^L)_{;L} + 2B (\phi_{I;K} \phi^L)_{;L} - g_{IK} L \right] \\ A \phi_{K;I}^K + B \phi_{I;K}^K &= -V' \phi_I \end{aligned}$$

contains two free parameters  $A$  and  $B$ .

Earlier the two possibilities  $A = \frac{1}{2}, B = 0$  and  $A = 0, B = \frac{1}{2}$  were considered in comparison, but separately [7]. As compared with the widely used scalar multiplet approach, in



the case of the vector order parameter the equations appear to be more simple, while their solutions are more general. In the case of two extra dimensions [7] the solutions have one more degree of parametric freedom, in addition to the arbitrary relation between  $A$  and  $B$ . This parametric freedom allows the existence of a braneworld without any restrictions of the fine-tuning type.

The general covariant equations derived above allow to investigate gravitational properties of more sophisticated longitudinal topological defects with arbitrary relation between  $A$  and  $B$ . A particular case  $A = -B$  is worth attention. In view of (12) the difference between the two cases  $A$  and  $B$  is connected with the curvature of space-time. It vanishes if  $R_{IK} = 0$ . Such an exotic topological defect has no analog in a flat world.

### B. Antisymmetric covariant derivative. Transverse vector field

In what follows I apply the general covariant equations to the case

$$A = B = 0, \quad C \neq 0. \quad (21)$$

and present a new braneworld solution with transverse vector field as the order parameter.

In the case (21) the set of equations (9), (19), and (16) reduces to

$$R_{IK} - \frac{1}{2}g_{IK}R = \kappa^2 [C (4F_I^L F_{LK} - g_{IK}F_{MN}F^{MN}) - 2V'\phi_K\phi_I + g_{IK}V], \quad (22)$$

$$CF_{I;L}^L = V'\phi_I. \quad (23)$$

Antisymmetric tensor  $F_{IK}$  is invariant against adding a gradient  $\psi_{;I}$  of any scalar  $\psi$  to  $\phi_I$  (gauge invariance):

$$\tilde{F}_{IK} = F_{IK} + \frac{1}{2}(\psi_{;I;K} - \psi_{;K;I}) = F_{IK} - \frac{1}{2}(\Gamma_{IK}^L - \Gamma_{KI}^L)\psi_{;L} = F_{IK}.$$

The set (22)-(23) is not gauge invariant because the order parameter (“vector-potential”)  $\phi_I$  enters the equations directly, and not only via the covariant derivatives (“forces”)  $F_{IK}$ . Only if  $V = V_0 = \text{const}$ ,  $V' = 0$  the equations (22)-(23) reduce to the gauge invariant set of Einstein-Maxwell equations (with no charges). In this case the term  $\kappa^2 g_{IK}V_0$  plays the role of a cosmological constant  $\Lambda = \kappa^2 V_0$ .

In the space-time with metric (20)

$$F_{IK} = \frac{1}{2}(\delta_{Kd_0}\phi'_I - \delta_{Id_0}\phi'_K). \quad (24)$$

For a longitudinal vector  $\phi_I = \phi \delta_{Id_0}$  the antisymmetric covariant derivative (24) is zero. A nonzero  $F_{IK}$  exists if  $\phi_I$  is a transverse vector. It should be directed along and depend on different coordinates. In the case of two extra dimensions  $(x^{d_0}, \varphi)$  the vector  $\phi_I$  can be directed azimuthally,

$$\phi_I = \phi \delta_{I\varphi}, \quad (25)$$

provided that all functions depend on  $x^{d_0} = l$ .

The Einstein equations (22), written in the form  $R_{IK} = \kappa^2 \left( T_{IK} - \frac{1}{d_0} g_{IK} T \right)$ , are

$$\begin{aligned} \gamma'' + \gamma' (d_0 \gamma' + \beta') &= -\frac{\kappa^2}{d_0} (C e^{-2\beta} \phi'^2 + 2V + 2V' e^{-2\beta} \phi^2) \\ - (d_0 \gamma'' + \beta'' + d_0 \gamma'^2 + \beta'^2) &= \frac{\kappa^2}{d_0} [-C (d_0 - 1) e^{-2\beta} \phi'^2 + 2V + 2V' e^{-2\beta} \phi^2] \\ \beta'' + \beta' (d_0 \gamma' + \beta') &= \frac{\kappa^2}{d_0} [C (d_0 - 1) e^{-2\beta} \phi'^2 - 2V + 2V' (d_0 - 1) e^{-2\beta} \phi^2]. \end{aligned} \quad (26)$$

The prime ' denotes the derivative  $\frac{d}{dl} : \gamma' = \frac{d\gamma}{dl}, \beta' = \frac{d\beta}{dl}, \phi' = \frac{d\phi}{dl}$ , except  $V' = \frac{dV}{d(\phi_K \phi^K)}$  (7).  $\gamma(l)$  enters the set (26) only via the derivatives  $\gamma'$  and  $\gamma''$ . The set (26) is of the fourth order with respect to unknowns  $\gamma', \beta$ , and  $\phi$ . The independent variable  $l$  is a cyclic coordinate. Excluding the second derivatives  $\beta''$  and  $\gamma''$ , we get a relation

$$\gamma' \beta' + \kappa^2 \left( \frac{C}{2d_0} e^{-2\beta} \phi'^2 + \frac{V}{d_0} \right) + \frac{d_0 - 1}{2} \gamma'^2 = 0, \quad (27)$$

which can be considered as a first integral of the system (26).

The vector field equation (23) reduces to

$$\phi'' + (d_0 \gamma' - \beta') \phi' = \frac{2}{C} V' \phi. \quad (28)$$

If the influence of matter on the brane is neglected, then there is no physical reason for singularities. For the metric (20) the Riemann curvature tensor is regular if the combinations

$$\gamma', \quad \gamma'' + \gamma'^2, \quad \beta' \gamma', \quad \beta'' + \beta'^2$$

are finite [6]. The finiteness of  $\beta'' + \beta'^2$  ensures the correct  $(= 2\pi)$  circumference-to-radius ratio at  $l \rightarrow 0$ . It excludes the curvature singularity in the center  $l = 0$ , which is a singular point of the cylindrical coordinate system. At  $l \rightarrow 0$  we have  $\beta' = \frac{1}{l} + O(l)$ . Finiteness of  $\beta' \gamma'$  dictates  $\gamma'(0) = 0$ . As far as the left hand sides of the equations (26) are finite, the combinations  $e^{-2\beta} \phi'^2 = \frac{\phi'^2}{r^2}$  and  $e^{-2\beta} \phi^2 = \frac{\phi^2}{r^2}$  should also be finite. Hence, for the transverse vector field at  $l \rightarrow 0$  we have

$$\beta' = \frac{1}{l}, \quad \gamma' = \gamma_0'' l, \quad \phi' = \phi_0'' l, \quad \phi = \frac{1}{2} \phi_0'' l^2, \quad l \rightarrow 0. \quad (29)$$

Only one of the two constants  $\gamma_0''$  and  $\phi_0''$  in (29) remains arbitrary. From the first integral (27) we find the connection between  $\gamma_0''$  and  $\phi_0''$  :

$$\gamma_0'' + \frac{\kappa^2}{2d_0} (C\phi_0''^2 + 2V_0) = 0. \quad (30)$$

Here  $V_0$  is the value of  $V$  at  $l = 0$ . In view of (29)  $\phi_K \phi^K = \frac{\phi^2}{r^2} = 0$  at  $l = 0$ , and  $V_0 = V(0)$ . The solutions with the transverse vector field and with the longitudinal one are different. In case of the longitudinal field, see [7], we had  $\phi = \phi_0' l$  at  $l \rightarrow 0$ .

For the regular solutions, spreading over the whole interval  $0 \leq l < \infty$ , we find from the Einstein equations (26) the following limiting value for both  $\beta_\infty'$  and  $\gamma_\infty'$  at  $l \rightarrow \infty$  :

$$\beta_\infty' = \gamma_\infty' = \sqrt{-\frac{2\kappa^2 V_\infty}{d_0(d_0+1)}}.$$

Here  $V_\infty$  is the limit of  $V$  at  $l \rightarrow \infty$ . For the solutions with infinitely growing  $r(l) \sim e^{\beta_\infty' l}$  the scalar  $\phi_K \phi^K = \frac{\phi^2}{r^2} \rightarrow 0$  at  $l \rightarrow \infty$ , and  $V_\infty$  coincides with  $V_0 = V(0)$ . The necessary condition of existence of the regular solutions, terminating at  $l = \infty$ , is  $V_\infty = V_0 < 0$ .

If  $\phi_0'' = 0$ , then the field equation (28) has a trivial solution  $\phi \equiv 0$ , corresponding to the state of unbroken symmetry. In the broken symmetry state  $\phi_0'' \neq 0$  the order parameter is not zero. However in the close vicinity of the phase transition  $\phi(l)$  is very small, and its influence on the metric is negligible. In the vicinity of phase transition, where  $\phi_0'' \ll 1$ , the equations (26) get so simplified

$$\gamma'' + \frac{d_0 + 1}{2} \gamma'^2 + \frac{\kappa^2 V_0}{d_0} = 0, \quad \beta' - (\ln \gamma')' - \gamma' = 0,$$

that for  $\gamma'$  and  $\beta'$  there is the analytical solution

$$\gamma'(l) = \gamma_\infty' \tanh\left(\frac{d_0 + 1}{2} \gamma_\infty' l\right), \quad \beta'(l) = \gamma_\infty' \left[ \frac{(d_0 + 1)}{\sinh[(d_0 + 1) \gamma_\infty' l]} + \tanh\left(\frac{d_0 + 1}{2} \gamma_\infty' l\right) \right]. \quad (31)$$

In the vicinity of phase transition the field  $\phi(l)$  obeys the equation (28) with  $\gamma'$  and  $\beta'$  from (31) and constant  $V' = V'(0) \equiv V_0'$ . If  $V_0'$  is zero, then  $\phi(l)$  is a growing function terminating at  $\phi_{inf} = \frac{2\phi_0''}{(d_0^2 - 1)\gamma_\infty'^2}$ , see red line in Figure 3 :

$$\phi(l) = \frac{2\phi_0''}{(d_0^2 - 1)\gamma_\infty'^2} \left\{ 1 - \left[ \cosh\left(\frac{d_0 + 1}{2} \gamma_\infty' l\right) \right]^{-2 \frac{d_0 - 1}{d_0 + 1}} \right\}, \quad V_0' = 0.$$

If  $V_0' \neq 0$ , then the asymptotic behavior of  $\phi(l)$  is determined by the equation

$$\phi'' + (d_0 - 1) \gamma_\infty' \phi' - \frac{2V_0'}{C} \phi = 0.$$

Its regular solutions, oscillating and vanishing at  $l \rightarrow \infty$ , (blue curves in Figure 3) exist only if  $V_0'/C < 0$ . Actually  $V_0' = 0$  is the point of phase transition.

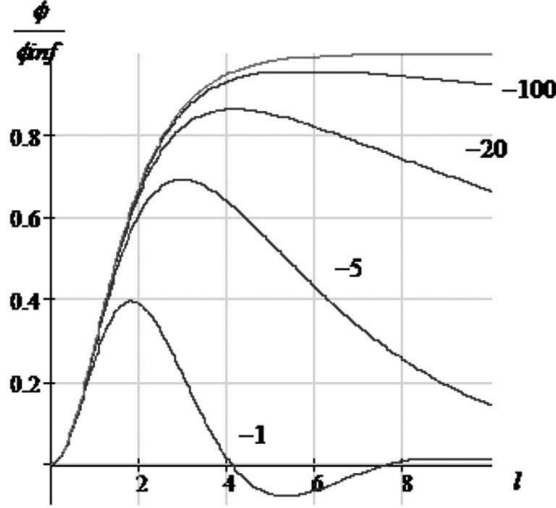


FIG. 3: Order parameter in the vicinity of phase transition. Red line – analytical solution in the limit  $V'_0/C = 0$ . Blue curves are numerical solutions for the pointed values of  $C/V'_0$ .

### VIII. NEUTRAL QUANTUM PARTICLE IN THE BRANEWORLD METRIC

A neutral spinless quantum particle is described by a scalar wave function  $\chi$  with the Lagrangian

$$L_\chi = \frac{1}{2}g^{AB}\chi_{,B}^*\chi_{,A} - \frac{1}{2}m_0^2\chi^*\chi. \quad (32)$$

In the uniform bulk (while the symmetry is not broken) it is a free particle in the  $D$ -dimensional space-time with mass  $m_0$  and spin zero. In the broken symmetry space-time with metric (20) it satisfies the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}g^{AB}\chi_{,A})_{,B} + m_0^2\chi = 0.$$

All coordinates except  $x^{d_0} = l$  are cyclic variables, and the conjugate momenta are quantum numbers. The wave function in a quantum state is

$$\chi(x^A) = X(l) \exp(-ip_\mu x^\mu + in\varphi), \quad (33)$$

where  $p_\mu = (E, \mathbf{p})$  is the  $d_0$ -momentum within the brane, and  $n$  is the integer angular momentum conjugate to the circular extradimensional coordinate  $\varphi$ .  $X(l)$  satisfies the equation [6]

$$X'' + WX' + (p^2 e^{-2\gamma} - n^2 e^{-2\beta} - m_0^2)X = 0. \quad (34)$$

The eigenvalues of  $p^2 = E^2 - \mathbf{p}^2$  compose the spectrum of squared masses, as observed in the brane. Quantum number  $n$  is the integer proper angular momentum of the particle,

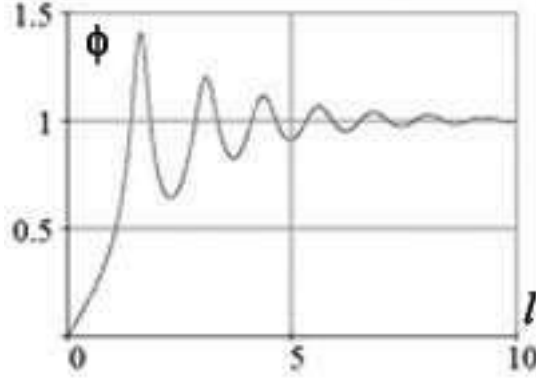


FIG. 4: A particular solution with oscillating order parameter  $\phi(l)$ . See details in [7].

conjugate to the extra-dimensional cyclic azimuthal coordinate  $\varphi$ . From the point of view of the observer in the brane it is the internal momentum, identical to the spin of the particle.

The equation (34) takes the form of the Schrodinger equation

$$y_{xx} + [p^2 - V_g(x)] y = 0 \quad (35)$$

after the substitution

$$dl = e^\gamma dx, \quad X(l) = y(x) / \sqrt{f(x)}, \quad f(x) = \exp \left\{ -\frac{1}{2} [(d_0 - 1) \gamma + \beta] \right\}.$$

The gravitational potential

$$V_g(x) = e^{2\gamma} (e^{-2\beta} n^2 + m_0^2) + \frac{1}{2} \frac{1}{\sqrt{f}} \frac{d}{dx} \left( \frac{1}{f^{1/2}} \frac{df}{dx} \right) \quad (36)$$

determines the trapping properties of particles to the brane.

Among the regular solutions there are those with damping oscillations of the order parameter. The oscillations of the order parameter  $\phi(l)$ , see an example in Fig.4, induce the oscillations of the gravitational potential (36). The gravitational potential can have many points of minimum, see Fig.5.

The length scale is an arbitrary parameter of the theory. If the scale length is extremely small, all points of minimum are located within one common brane. In the spirit of Kalutza-Kline the points of minimum are beyond the resolution of modern devices.

Low energy particles can be trapped within the points of minimum of the potential (36). Identical in the bulk neutral spin-less particles, being trapped in the different minimum points, acquire different masses and angular momenta. If the scale length is extremely small, then for the observer within the brane they appear as different particles with integer spins.

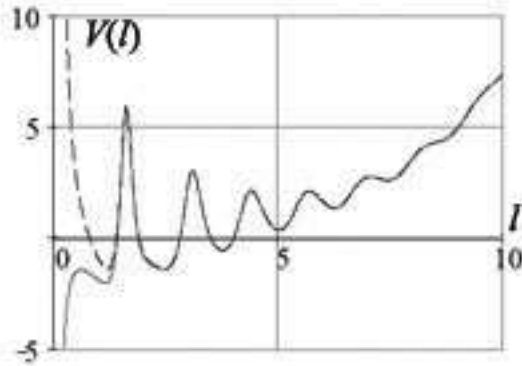


FIG. 5: Gravitational potential  $V_g(l)$  for the same set of the parameters as in Fig.4. The red curve corresponds to the angular momentum  $n = 0$ , and the dashed blue one – to  $n = 1$ .

Most elementary particles have half-integer spins. The simple cases, considered above, when the vector order parameter depends on only one coordinate, can not connect the origin of half-integer spins with extra-dimensional angular momenta. The question, if more sophisticated cases of spontaneous symmetry breaking can demonstrate the extra-dimensional origin of half-integer spins, remains open.

## IX. CONCLUDING REMARKS

The advantages of the vector order parameter over a scalar multiplet become evident from the table.

Comparison of scalar multiplet and vector field solutions				
Property	Multiplet	Vector, $A \neq 0$	Vector, $B \neq 0$	Vector, $C \neq 0$
Order of Einstein equations	4	3		4
Number of free parameters	$n_V$	$n_V + 1$		
Fine tuning	in some cases	no need		
$r$ at $l \rightarrow l_{max}$	$0, r_m, \infty$	$r \rightarrow \infty$ at $l \rightarrow \infty$		
Matter trapping	yes			
Derivations are	easy	wearing		easy
Equations are	simple	most simple	more simple	
Solutions are	less general	more general		
Cosmological constant	negative			
Strength of gravitational field	arbitrary		restricted from above	arbitrary
Order parameter $\phi$ at $l \rightarrow 0$	$\phi \sim l$			$\phi \sim l^2$
$n_V$ is the number of parameters of the potential $V(\phi^K \phi_K)$				

Besides the forthcoming detailed analytical and numerical analysis of extra-dimensional

transverse vector field, I would like to mention the following possible applications of the derived general covariant equations:

- revision of previous results with scalar multiplet models
- possibility of extra-dimensional origin of particles with half-integer, and not only with integer spins
- dynamics of phase transitions, including the braneworld formation
- search for observational consequences of extra dimensions.

The covariant equations with vector order parameter are suitable for, but not limited to the mentioned problems.

- 
- [1] L.D.Landau. Phys.Zs. Sowjet., **11**, 26, (1937). (Collected papers, “Nauka”, Moscow 1969, vol.1, paper 28)
  - [2] D.A.Kirzhnits. JETP Lett., **15**, 745, (1972)
  - [3] A.Vilenkin, E.P.S.Shellard. Cosmic Strings and Other Topological Defects. Cambridge Univ. Press, 1994
  - [4] A.M.Polyakov. JETP Lett., **20**, 194, (1974)
  - [5] G.’t Hooft. Nucl. Phys., **B79**, 276, (1974)
  - [6] K.A.Bronnikov and B.E.Meierovich. Zh.Eksp. Teor. Fiz. **133**, No. 2, pp. 293-312 (2008)
  - [7] B.E.Meierovich. PHYSICAL REVIEW **D 79**, 104027, (2009)
  - [8] B.E.Meierovich. PHYSICAL REVIEW **D 82**, 024004, (2010)